Curtin University, Semester 1, 2022
ECON 4002 (Dr. Lei Pan)
Problem Set 1 Solution
Due Friday, April 1st at 5:00pm AWST

Question 1. [30 marks] Solow-Swan model

Suppose the production function is Cobb-Douglas. (i.e. $Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$)

(a) [10 marks] Find expressions for k^*, y^* and c^* as functions of the parameters of the models, s, n, δ, g and α.

Answer The transition equation describing the evolution of capital stock per unit of effective labour is given by:

$$k_{t+1} = \frac{(1-\delta)}{(1+n)(1+g)} k_t + \frac{s}{(1+n)(1+g)} f(k_t)$$

To find the steady state value of k_t (i.e. k^*), let $k_{t+1} = k_t = k^*$ in the above transition equation, and rearrange items, we have:

$$\begin{aligned} sf(k^*) &= (n+g+\delta)k^* \\ s(k^*)^\alpha &= (n+g+\delta)k^* \\ k^* &= (\frac{s}{n+g+\delta})^{\frac{1}{1-\alpha}} \qquad (1) \end{aligned}$$

Then the balanced-growth-path value of output per unit of effective labour is:

$$y^* = (k^*)^\alpha = (\frac{s}{n+g+\delta})^{\frac{\alpha}{1-\alpha}}$$

Consumption per unit of effective labour on the balanced growth path is given by:

$$c^* = (1-s)(\frac{s}{n+g+\delta})^{\frac{\alpha}{1-\alpha}}$$

(b) [10 marks] What is the golden-rule value of k?

Answer By definition, the golden-rule level of capital stock maximises the steady state value of consumption per unit of effective labour.

$$i_t = \frac{I_t}{A_t L_t} = \frac{K_{t+1} - (1-\delta)K_t}{A_t L_t} = k_{t+1}(1+g+n) - (1-\delta)k_t$$

Assuming $gn = 0$. Thus, on a balanced growth path, investment per unit of effective labour is given by:

$$i = (g+n+\delta)k$$

This is the amount of investment per unit of effective labour that is needed to keep capital stock per unit of effective labour steady along a balanced growth path.

Therefore, the golden-rule level of k maximises $k^\alpha - (n+g+\delta)k$. The first order condition is given by:

$$\alpha k_{GR}^{\alpha-1} = (n+g+\delta)$$
$$k_{GR} = (\frac{\alpha}{n+g+\delta})^{\frac{1}{1-\alpha}} \tag{2}$$

(c) [5 marks] What saving rate is needed to yield the golden-rule capital stock?

Answer To obtain the saving rate that yields the golden-rule level of capital stock, equate Eq.(1) and (2):

$$(\frac{s}{n+g+\delta})^{\frac{1}{1-\alpha}} = (\frac{\alpha}{n+g+\delta})^{\frac{1}{1-\alpha}}$$

which can be simplified to:

$$s_{GR} = \alpha$$

With a Cobb-Douglas production function, the saving rate required to reach the golden rule capital stock is equal to the elasticity of output with respect to capital or capital's share in output (if capital earns its marginal product).

(d) [5 marks] Describe how, if at all, each of the following developments affects the steady state capital per unit of effective labour in the Solow-Swan model (*Hint: Think about how lines $sf(k)$ and $(n+g+\delta)$ would shift*).

(i) [3 marks] The rate of technological progress rises.

Answer A rise in the rate of technological progress, g, makes the break-even investment line steeper. The actual investment curve is unaffected. Therefore, k^* falls. (Note: That k^* falls does not mean that output per worker falls since output per worker $\frac{Y_t}{L_t} = A_t f(k^*)$. Despite that k^* falls, A_t is now growing at a faster rate.)

(ii) [2 marks] The saving rate rises.

Answer No change in break-even investment line but saving line go up, therefore, k^* and y^* increases.

Question 2. **[20 marks] Consumer maximisation problem (2 periods)**

Solve the following optimisation problem of the representative consumer:

$$\max_{c_1,c_2,k_2} V_1 = \max_{c_1,c_2,k_2} U(c_1) + \beta U(c_2)$$

subject to

$$\begin{aligned} c_1 + k_2 &= F(k_1) + (1-\delta)k_1 \\ c_2 &= F(k_2) + (1-\delta)k_2 \end{aligned}$$

Period utility is given by:

$$U(c_t) = \log c_t$$

and output is given by:

$$y_t = F(k_t) = k_t^\alpha$$

where $t = 1, 2$.

(a) [15 marks] Derive the Euler equation for the specific utility- and production function by employing the Lagrange approach.

Answer Recall the capital accumulation equation is given by:

$$k_{t+1} = k_t + i_t - \delta k_t$$

The national accounting identity (for a closed economy) is:

$$y_t = c_t + i_t$$

Putting both together yields:

$$\begin{aligned} y_t &= c_t + k_{t+1} - (1-\delta)k_t \\ c_t + k_{t+1} &= y_t + (1-\delta)k_t \end{aligned}$$

The above equation indicates that consumer can shift resources intertemporally by investing in the capital stock.

The objective function reads

$$\max_{c_1,c_2,k_2} V_1 = \max_{c_1,c_2,k_2} \log c_1 + \beta \log c_2$$

This means that the household chooses consumption in period 1 (c_1) and period 2 (c_2) in order to maximise its lifetime utility function which we call V_1. In this problem the lifetime of the household is two periods only. One can think of two periods where the household is young and old respectively. $0 < \beta < 1$ is the discount factor. It represents impatience of the household.

In our particular case the objective function is logarithmic. We usually assume that the objective function is concave. The household now chooses in every period of life how much to consume and how much to save for the next period. There is a trade-off between consumption today (period 1) and tomorrow (period 2). The household could consume more (save less) today but then it has to consume less tomorrow (because of the low savings). The trade-off comes by the budget constraints.

We maximise the lifetime utility function V_1 with respect to the budget constraints of the household. If there would not be any constraint the household could simply maximise its lifetime utility by consuming an infinite amount of c_1 and c_2.

Set up the Lagrangian equation:

$$\mathcal{L} = \log c_1 + \beta \log c_2 + \lambda_1[k_1^\alpha + (1-\delta)k_1 - c_1 - k_2] + \lambda_2[k_2^\alpha + (1-\delta)k_2 - c_2]$$

F.O.C with choice variables

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} - \lambda_1 = 0 \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \beta \frac{1}{c_2} - \lambda_2 = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial k_2} = -\lambda_1 + \lambda_2[\alpha k_2^{\alpha-1} + (1-\delta)] = 0 \tag{5}$$

Rearranging Eq.(5) yields:

$$\lambda_1 = \lambda_2[\alpha k_2^{\alpha-1} + (1-\delta)] \tag{6}$$

Substituting Eq.(3) and (4) into (6), gives:

$$\frac{1}{c_1} = \beta \frac{1}{c_2}[1 + \alpha k_2^{\alpha-1} - \delta] \tag{7}$$

Eq.(7) is known as the Euler equation.

(b) [5 marks] What does your result say about the consumption path of the consumer?

Answer The Euler equation describes the optimal consumption path. It equates the marginal consumption today with the marginal consumption tomorrow (from saving) discounted by β.

Note that $1 + \alpha k_2^{\alpha-1} - \delta$ can be interpreted as an interest rate. In the optimum the consumer cannot improve her utility by shifting consumption intertemporally.

We can rewrite Eq.(7) to the following:

$$\frac{U'(c_1)}{\beta U'(c_2)} = 1 + \alpha k_{t+1}^{\alpha-1} - \delta \tag{8}$$

Here we equate the marginal rate of substitution between consumption today and tomorrow (left hand side) and the marginal rate of transformation (right hand side).

Question 3. **[50 marks] Social security in the Diamond model**

Consider a Diamond economy where g is zero, production function is Cobb-Douglas, and utility is logarithmic.

Pay-as-you-go social security

Suppose the government taxes each young individual an amount τ and uses the proceeds to pay benefits to old individuals; thus each old person receives $(1+n)\tau$ (government has a balanced budget).

(a) [20 marks] Write down the individual's utility maximisation problem, derive the first order condition, solve for s_t and find $\frac{\partial s_t}{\partial \tau}$, $\frac{\partial c_{1t}}{\partial \tau}$ and $\frac{\partial c_{2t+1}}{\partial \tau}$. Discuss how s_t, c_{1t} and c_{2t+1} change with τ.

Answer The individual's utility maximisation problem is given by:

$$\max\ u(c_{1t}, c_{2t+1}) = \log(c_{1t}) + \beta \log(c_{2t+1})$$

$$c_{1t} + s_t = Aw_t - \tau \tag{9}$$

$$c_{2t+1} = (1 + r_{t+1})s_t + (1 + n)\tau \tag{10}$$

The only choice variable is s_t. The F.O.C is given by:

$$\begin{aligned}\frac{\partial U}{\partial s_t} &= \frac{1}{c_{1t}}(-1) + \beta\frac{1}{c_{2t+1}}(1 + r_{t+1}) = 0\\ &= -\frac{1}{Aw_t - \tau - s_t} + \beta\frac{(1 + r_{t+1})}{(1 + r_{t+1})s_t + (1 + n)\tau} = 0\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{Aw_t - \tau - s_t} &= \beta\frac{(1 + r_{t+1})}{(1 + r_{t+1})s_t + (1 + n)\tau}\\ \beta(1 + r_{t+1})(Aw_t - \tau - s_t) &= (1 + r_{t+1})s_t + (1 + n)\tau\\ (1 + r_{t+1})s_t + \beta(1 + r_{t+1})s_t &= \beta(1 + r_{t+1})(Aw_t - \tau) - (1 + n)\tau\\ (1 + r_{t+1})(1 + \beta)s_t &= \beta(1 + r_{t+1})(Aw_t - \tau) - (1 + n)\tau\end{aligned}$$

Hence

$$s_t = \frac{\beta}{1 + \beta}(Aw_t - \tau) - \frac{(1 + n)}{(1 + r_{t+1})(1 + \beta)}\tau$$

Now we can do the comparative statics.

$$\frac{\partial s_t}{\partial \tau} = \frac{-\beta}{1 + \beta} - \frac{1 + n}{(1 + r_{t+1})(1 + \beta)} = -\frac{\beta(1 + r_{t+1}) + (1 + n)}{(1 + r_{t+1})(1 + \beta)} < 0$$

Note that $\frac{\partial s_t}{\partial \tau} = -1$ if $r_{t+1} = n$; $\frac{\partial s_t}{\partial \tau} < -1$ if $r_{t+1} < n$; and $\frac{\partial s_t}{\partial \tau} > -1$ if $r_{t+1} > n$. In other words, if $r_{t+1} = n$, saving is reduced one-for-one by the social security tax. If $r_{t+1} < n$, saving falls more than one-for-one. If $r_{t+1} > n$, saving falls less than one-for-one.

Since $c_{1t} = Aw_t - \tau - s_t$ and $c_{2t+1} = (1 + r_{t+1})s_t + (1 + n)\tau$, we therefore have:

$$\begin{aligned}
\frac{\partial c_{1t}}{\partial \tau} &= -\frac{\partial s_t}{\partial \tau} - 1 \\
&= \frac{\beta(1 + r_{t+1}) + (1 + n)}{(1 + r_{t+1})(1 + \beta)} - 1 \\
&= \frac{\beta(1 + r_{t+1}) + (1 + n) - (1 + r_{t+1})(1 + \beta)}{(1 + r_{t+1})(1 + \beta)} \\
&= \frac{n - r_{t+1}}{(1 + r_{t+1})(1 + \beta)} \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial c_{2t+1}}{\partial \tau} &= (1 + r_{t+1})\frac{\partial s_t}{\partial \tau} + (1 + n) \\
&= -(1 + r_{t+1})\frac{\beta(1 + r_{t+1}) + (1 + n)}{(1 + r_{t+1})(1 + \beta)} + (1 + n) \\
&= \frac{-\beta(1 + r_{t+1}) - (1 + n) + (1 + n)(1 + \beta)}{1 + \beta} \\
&= \frac{-\beta(1 + r_{t+1}) + (1 + n)\beta}{1 + \beta} \\
&= \frac{\beta(n - r_{t+1})}{1 + \beta} \qquad (12)
\end{aligned}$$

From Eq.(11) and (12), the sign of $\frac{\partial c_{1t}}{\partial \tau}$ and $\frac{\partial c_{2t+1}}{\partial \tau}$ agrees with $n - r_{t+1}$: if $r_{t+1} < n$, marginal increase in τ would increase consumption in both periods; if $r_{t+1} > n$, a marginal increase in τ would decrease consumption in both periods.

To see the intuition, consider the lifetime income of the individual, which can be seen clearly by deriving the lifetime budget constraint of the individual. From the second period budget constraint (10), $s_t = \frac{c_{2t+1} - (1+n)\tau}{(1+r_{t+1})}$. Substituting this into the first budget constraint (9) yields:

$$c_{1t} + \frac{c_{2t+1}}{(1 + r_{t+1})} = Aw_t + \frac{n - r_{t+1}}{(1 + r_{t+1})}\tau \qquad (13)$$

where the left hand side of (13) is the present value of the lifetime consumption, while the right hand side is the present value of lifetime income, which includes two parts: the first part is the labour income when young and the second part is the present value of the net benefit from the social security tax (By paying a tax of τ at young age, the individual gets $(1 + n)\tau$ at old age. If the individual saves τ by himself rather than paying the tax, he would get $(1 + r_{t+1})\tau$ at old age. So the net benefit at old age from paying a tax of τ when young is $(n - r_{t+1})\tau$, and dividing by $(1 + r_{t+1})$ gives its present value). It is clear that a marginal increase in τ would lead to a higher lifetime income if $r_{t+1} < n$, which would increase consumption in both periods. Note that n is the social return on tax, while r_{t+1} is the market return on private saving. If the market return on private saving is lower than the social return on tax, the gain from one extra unit of tax exceeds the gain from saving it.

(b) [10 marks] Write k_{t+1} as a function of k_t. Is k^* bigger or smaller than the old value of k^* when the social security is absent, i.e. if $\tau = 0$ (*Hint: You do not need to solve for* k^*. *Just compare the transition equations in two cases to see whether introducing the social security shifts the transition equation up or down. Then you will find out whether* k^* *is bigger or smaller*).

Answer Part (a) characterises the individuals' behavior. To characterise the equilibrium, we also need to consider the firms' behavior. Firms hire labour and rent capital at competitive markets and produce output. Since markets are competitive, capital and labour earn their marginal products, i.e.

$$r_t = f'(k_t) = \alpha k_t^{\alpha-1} \tag{14}$$
$$w_t = f(k_t) - k_t f'(k_t) = (1-\alpha)k_t^{\alpha} \tag{15}$$

where k_t is defined as the capital stock per unit of effective labour. Since capital is formed by the savings of young individuals, the aggregate capital stock at the beginning of period $t+1$ is given by:

$$K_{t+1} = L_t s_t \tag{16}$$

Dividing both sides by AL_{t+1}, get:

$$\begin{aligned} k_{t+1} &= \frac{1}{1+n}\frac{s_t}{A} \\ &= \frac{1}{1+n}\frac{\left(\frac{\beta}{1+\beta}(Aw_t - \tau) - \frac{(1+n)}{(1+r_{t+1})(1+\beta)}\tau\right)}{A} \end{aligned}$$

Plugging Eq.(14) and (15) into the above equation, we obtain:

$$k_{t+1} = \frac{\beta}{(1+\beta)(1+n)}(1-\alpha)k_t^{\alpha} - \frac{\left[\frac{\beta}{(1+\beta)(1+n)} + \frac{1}{(1+\beta)(1+\alpha k_{t+1}^{\alpha-1})}\right]\tau}{A} \tag{17}$$

The dynamic of k without the social security is

$$k_{t+1} = \frac{\beta}{(1+\beta)(1+n)}(1-\alpha)k_t^{\alpha} \tag{18}$$

Comparing Eq.(17) with (18), we see that with the social security k_{t+1} curve shifts down (for any given k_t, the value of k_{t+1} is lower) such that k^* is smaller.

Fully funded social security

Suppose the government taxes each young person an amount τ and uses the proceeds to purchase capital. Individuals born at t therefore receive $(1+r_{t+1})\tau$ when they are old.

(c) [10 marks] Write down the individual's utility maximisation problem, derive the first order condition and solve for s_t. Find $\frac{\partial s_t}{\partial \tau}$ and explain the effect of τ on private saving.

Answer Household's optimisation problem:

$$\max \ u(c_{1t}, c_{2t+1}) = \log(c_{1t}) + \beta \log(c_{2t+1})$$

$$c_{1t} + s_t = Aw_t - \tau \tag{19}$$

$$c_{2t+1} = (1 + r_{t+1})s_t + (1 + r_{t+1})\tau \tag{20}$$

Therefore, $s_t + \tau = \frac{c_{2t+1}}{(1+r_{t+1})}$. Substitute into Eq.(19) yields the individual's lifetime budget constraint

$$c_{1t} + \frac{c_{2t+1}}{(1 + r_{t+1})} = Aw_t \tag{21}$$

Note that this is the same budget constraint when social security is absent. Therefore, c_{1t}, c_{2t+1} stay the same as when social security is absent. Hence, $s_t + \tau = Aw_t - c_{1t}$ equals the original level of saving when social security is absent such that s_t equals the original saving minus the tax τ (we assume that τ is not greater than the amount of saving each young individual would have done in the absence of the tax). It is easy to work out that $s_t = \frac{\beta}{1+\beta}Aw_t - \tau$. Hence, $\frac{\partial s_t}{\partial \tau} = -1$, i.e. the social security tax causes a one-for-one reduction in private saving.

(d) [5 marks] Write k_{t+1} as a function of k_t. How does k^* differ from the old value of k^* when the social security is absent? And why?

Answer The capital stock in period $t+1$ will be equal to the sum of private saving of the young plus the total amount investment by the government (public saving). Hence,

$$K_{t+1} = L_t s_t + L_t \tau = \frac{\beta}{1 + \beta} Aw_t L_t$$

Dividing AL_{t+1} on both sides of the above equation, we have:

$$k_{t+1} = \frac{\beta}{(1 + n)(1 + \beta)}(1 - \alpha)k_t^{\alpha}$$

This is the same equation characterising the dynamics of k when the social security is absent. Therefore, the fully funded social security has no effect on the relationship between k_{t+1} and k_t so that k^* remains the same.

(e) [5 marks] Can the government use fully funded social security to improve on a decentralised equilibrium?

Answer Since there is no effect on the relationship between k_{t+1} and k_t, the balanced-growth-path value of k is the same as it was before the introduction of the fully funded social security system. The basic idea is that total investment and saving is still the same in each period; the government is simply doing some of the saving for the young. Since social security pays the same rate of return as private saving, individuals are indifferent as to who does the saving. Thus individuals offset one-for-one any saving the government does for them.